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ORIGINAL RESEARCH

Infinitely many solutions for fractional differential system via variational method

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Abstract In this paper we investigate a boundary value problem for a coupled nonlinear differential system of fractional order. Under appropriate hypotheses and by applying the critical point theorem, we obtain some new criteria to guarantee that the fractional differential system has infinitely many weak solutions. In addition, an example is given to illustrate the main results.

Keywords Infinitely many solutions · Fractional differential system · Variational method · Critical points

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1 Introduction

In this paper we are concerned with the existence and multiplicity of weak solutions for the following fractional differential system

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$$\begin{cases} {}_tD_T^{\alpha}(a(t)_0D_t^{\alpha}u(t)) = \lambda F_u(t,u(t),v(t)) + h_1(u(t)), & 0 < t < T, \\ {}_tD_T^{\beta}(b(t)_0D_t^{\beta}v(t)) = \lambda F_v(t,u(t),v(t)) + h_2(v(t)), & 0 < t < T, \\ u(0) = u(T) = 0, \quad v(0) = v(T) = 0, \end{cases} \tag{1.1}$$

where λ is a positive real parameter, $0 < \alpha, \beta \leq 1$, $a, b \in L^{\infty}[0,T]$ with $a_0 := \operatorname{ess\,inf}_{[0,T]} a(t) > 0$ and $b_0 := \operatorname{ess\,inf}_{[0,T]} b(t) > 0$, ${}_0D_t^{\gamma}$ and ${}_tD_T^{\gamma}$ denote the left and right Riemann–Liouville fractional derivatives of order γ respectively, and $F : [0,T] \times \mathbf{R^2} \to \mathbf{R}$ is a function such that $F(\cdot, x, y)$ is continuous in $[0,T]$ for every $(x,y) \in \mathbf{R^2}$ and $F(t,\cdot,\cdot)$ is a C^1 function in $\mathbf{R^2}$ for any $t \in [0,T]$, and F_u, F_v denote the partial derivative of F, with respect to u, v respectively. $h_1, h_2 : \mathbf{R} \to \mathbf{R}$ be two Lipschitz continuous functions with the Lipschitz constants $L_1, L_2 \geq 0$; i.e.,

$$|h_i(x_1) - h_i(x_2)| \leq L_i|x_1 - x_2|, \quad i = 1, 2 \tag{1.2}$$

for every $x_1, x_2 \in \mathbf{R}$, satisfying $h_i(0) = 0$, $i = 1, 2$.

Fractional differential equations have gained importance due to their numerous applications in various fields of science and engineering, such as fluid flow, diffusive transport akin to diffusion, rheology, probability, electrical networks, etc. For details, see [1–3]. Recently, a great deal of work has been done in the study of the existence and uniqueness of solutions to nonlinear fractional differential equations (see [1–10,31–33] and the references therein). Some classical tools have been used to investigate such problems in the literature, such as some fixed point theorems in cones, the coincidence degree theory of Mawhin, and the method of upper and lower solutions with the monotone iterative technique. The study of a coupled system of fractional order is also very significant because this kind of system can often occur in applications; (see [11–15]). Using the Krasnoselskii's fixed point theorem and the nonlinear alternative of Leray–Schauder theorem in a cone, Bai and Fang [11] studied the existence of a positive solution to singular coupled system of fractional order. By applying the Schauder fixed point theorem, Ahmad and Nieto [13] discussed the existence of a coupled differential system of fractional order with three-point boundary conditions.

On the other hand, critical point theory has been very useful in dealing with the existence and multiplicity of solutions for integer order differential equations with some boundary conditions. We refer readers to the books due to Mawhin and Willem[18], Schechter[19], and the papers [20–24] and the references therein. But until now, there are few works that deal with the fractional boundary value problems and boundary value systems via the variational methods; see [16,17,25–30]. Besides, it is worth mentioning that the fractional calculus of variations was introduced in [25]. By means of critical point theory, Jiao and Zhou [16] considered the following fractional boundary value problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (0,1]$, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann–Liouville fractional derivatives respectively. $F : [0,T] \times \mathbf{R^N} \to \mathbf{R}$ (with $N \geq 1$) is a suitable given function and $\nabla F(t,u)$ is the gradient of F with respect to u.

In [28], Bai investigated the following perturbed nonlinear fractional boundary value problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \lambda\, a(t)f(u(t)) + \mu g(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where $\alpha \in (0,1]$, λ, μ are non-negative parameters, $a : [0,T] \to \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ and $g : [0,T] \times \mathbf{R} \to \mathbf{R}$ are three continuous functions. By using a recent critical point theorem of Bonanno and Molica Bisci [21], the existence of infinitely many solutions for the problem (1.3) depending on two parameters are obtained.

In [15], By applying the critical point theorem due to Bonanno and Marano [22], we provided a new approach to studied the existence of weak solutions for the following fractional differential system

$$\begin{cases} {}_tD_T^{\alpha}(a(t){}_0D_t^{\alpha}u(t)) = \lambda\, F_u(t,u(t),v(t)), \quad 0 < t < T, \\ {}_tD_T^{\beta}(b(t){}_0D_t^{\beta}v(t)) = \lambda\, F_v(t,u(t),v(t)), \quad 0 < t < T, \\ u(0) = u(T) = 0, \quad v(0) = v(T) = 0. \end{cases} \tag{1.4}$$

The main result is as follows.

In this article, we need the following conditions.

(H0) $\frac{1}{2} < \alpha, \beta \le 1$.

(H1) $F : [0,T] \times \mathbf{R^2} \to \mathbf{R}$ be a function such that $F(\cdot,u,v)$ is continuous in $[0,T]$ for every $(u,v) \in \mathbf{R^2}$, $F(t,\cdot,\cdot)$ is a C^1 function in $\mathbf{R^2}$.

Put

$$M = \max\left\{ \frac{T^{2\alpha-1}}{(\Gamma(\alpha))^2 a_0 (2\alpha-1)}, \frac{T^{2\beta-1}}{(\Gamma(\beta))^2 b_0 (2\beta-1)} \right\},$$
$$L = \max\left\{ \frac{T^{2\alpha}}{(\Gamma(\alpha+1))^2 a_0}, \frac{T^{2\beta}}{(\Gamma(\beta+1))^2 b_0} \right\}. \tag{1.5}$$

Theorem 1.1 ([15, Theorem 3.1]) *Assume that $F(t,0,0) = 0$ for all $t \in [0,T]$ and (H0),(H1) hold. Furthermore suppose that there exist a constant $r > 0$ and a function $\omega = (\omega_1, \omega_2)$ such that*

(C1) $\|\omega_1\|_\alpha^2 + \|\omega_2\|_\beta^2 > 2r$;

(C2) $$\frac{\int_0^T \sup_{(\xi,\eta)\in\Omega(Mr)} F(t,\xi,\eta)dt}{r} < \frac{2\int_0^T F(t,\omega_1(t),\omega_2(t))dt}{\|\omega_1\|_\alpha^2 + \|\omega_2\|_\beta^2};$$

(C3) $$\liminf_{|\xi|\to+\infty, |\eta|\to+\infty} \frac{F(t,\xi,\eta)}{|\xi|^2 + |\eta|^2} < \frac{\int_0^T \sup_{(\xi,\eta)\in\Omega(Mr)} F(t,\xi,\eta)dt}{2Lr},$$

where $\Omega(Mr) = \{(\xi,\eta) \in \mathbf{R^2} : \frac{1}{2}|\xi|^2 + \frac{1}{2}|\eta|^2 \le Mr\}$.
Then, for each

$$\lambda \in \Lambda := \left] \frac{\|\omega_1\|_\alpha^2 + \|\omega_2\|_\beta^2}{2\int_0^T F(t,\omega_1(t),\omega_2(t))dt}, \frac{r}{\int_0^T \sup_{(\xi,\eta)\in\Omega(Mr)} F(t,\xi,\eta)dt} \right[,$$

the coupled system (1.4) has at least three distinct weak solutions.

Motivated by the above work, in this paper we devote to study the multiplicity of weak solutions of problem (1.1) via variational method. To our knowledge, the study of solutions for nonlinear fractional differential system using variational method has received considerably less attention. Our main contributions in this article include two aspects. Firstly, we successfully construct a suitable space and obtain a variational functional for problem (1.1), and consequently establish some new results for the existence of infinitely many solutions of problem (1.1) using the critical point theorem. Another contribution is that, through this work, we have successfully shown that the critical point theory is an effective approach to deal with the existence of solutions for fractional differential system. The rest of the article is organized as follows. In Sect. 2, some definitions and lemmas that will be useful for our main results are given. In Sect. 3, Several criteria for the existence of infinitely many weak solutions of the coupled system (1.1) are established and an example is presented to illustrate the main results.

2 Preliminaries

To construct appropriate function spaces and apply critical point theory to investigate the existence of solutions for problem (1.1), we need the following some basic notations and results which will be used in the proof of our main results.

Let $C_0^\infty([0,T],\mathbf{R^N})$ be the set of all functions $x \in C_0^\infty([0,T],\mathbf{R^N})$ with $x(0) = x(T) = 0$ and the norm

$$\|x\|_\infty = \max_{[0,T]} |x(t)|. \tag{2.1}$$

Denote the norm of the space $L^p([0,T],\mathbf{R^N})$ for $1 \le p < \infty$ by

$$\|x\|_{L^p} = \left(\int_0^T |x(s)|^p ds\right)^{1/p}.$$

The following lemma yields the boundedness of the Riemann–Liouville fractional integral operators from the space $L^p([0,T],\mathbf{R^N})$ to the space $L^p([0,T],\mathbf{R^N})$, where $1 \le p < \infty$.

Lemma 2.1 ([17]) *Let* $0 < \alpha \le 1$, $1 \le p < \infty$ *and* $f \in L^p([0,T],\mathbf{R^N})$. *Then*

$$\|{}_0D_\xi^{-\alpha} f\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p([0,t])}, \quad \text{for } \xi \in [0,t], t \in [0,T],$$

where ${}_0D_t^{-\alpha}$ *is left Riemann–Liouville fractional integral of order* α.

Definition 2.2 Let $0 < \alpha \le 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0,T],\mathbf{R})$, that is

$$E_0^\alpha = \overline{C_0^\infty([0,T],\mathbf{R})}$$

with respect to the weighted norm

$$\|u\|_\alpha = \left(\int_0^T a(t)|{}_0D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\right)^{1/2}, \quad \forall u \in E_0^\alpha. \tag{2.2}$$

Clearly, the fractional derivative space E_0^α is the space of functions $u \in L^2[0, T]$ having an α-order fractional derivative ${}_0D_t^\alpha u \in L^2[0, T]$ and $u(0) = u(T) = 0$. From [17, Proposition3.1], we know for $0 < \alpha \leq 1$, the space E_0^α is a reflexive and separable Banach space.

Lemma 2.3 ([15]) *Let* $0 < \alpha \leq 1$. *For any* $u \in E_0^\alpha$, *we have*

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)\sqrt{a_0}} \left(\int_0^T a(t)|{}_0D_t^\alpha u(t)|^2 dt\right)^{1/2}. \tag{2.3}$$

Moreover, if $\alpha > \frac{1}{2}$, *then*

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}} \left(\int_0^T a(t)|{}_0D_t^\alpha u(t)|^2\right)^{1/2}. \tag{2.4}$$

By (2.3), we can consider E_0^α *with respect to the norm*

$$\|u\|_\alpha = \left(\int_0^T a(t)|{}_0D_t^\alpha u(t)|^2 dt\right)^{1/2}, \quad \forall u \in E_0^\alpha, \tag{2.5}$$

which is equivalent to (2.2).

For further purpose we will consider the fractional derivative space E_0^β *be defined by the closure of* $C_0^\infty([0, T], \mathbf{R})$ *with respect to the weighted norm*

$$\|v\|_\beta = \left(\int_0^T b(t)|{}_0D_t^\beta v(t)|^2 dt + \int_0^T |v(t)|^2 dt\right)^{1/2}, \quad \forall v \in E_0^\beta. \tag{2.6}$$

Lemma 2.4 ([15]) *Let* $0 < \beta \leq 1$. *For any* $u \in E_0^\beta$, *one has*

$$\|v\|_{L^2} \leq \frac{T^\beta}{\Gamma(\beta+1)\sqrt{b_0}} \left(\int_0^T b(t)|{}_0D_t^\beta v(t)|^2 dt\right)^{1/2}. \tag{2.7}$$

Moreover, if $\beta > \frac{1}{2}$, *then*

$$\|v\|_\infty \leq \frac{T^{\beta-\frac{1}{2}}}{\Gamma(\beta)\sqrt{b_0(2\beta-1)}} \left(\int_0^T b(t)|{}_0D_t^\beta v(t)|^2\right)^{1/2}. \tag{2.8}$$

Obviously, the space E_0^β *is also a reflexive and separable Banach space with the morm*

$$\|v\|_\beta = \left(\int_0^T b(t)|{}_0D_t^\beta v(t)|^2 dt\right)^{1/2}, \quad \forall v \in E_0^\beta, \tag{2.9}$$

which is equivalent to (2.6).

Similarly to [17, proposition 3.3], we have the following property of the fractional derivative space E_0^α *(or* E_0^β*).*

Lemma 2.5 *Assume that* $\frac{1}{2} < \alpha \le 1$ *and the sequence* $\{u_n\}$ *converges weakly to* u *in* E_0^α*, i.e.* $u_n \rightharpoonup u$*. Then* $\{u_n\}$ *converges strongly to* u *in* $C([0,T],\mathbf{R})$*, i.e.* $\|u_n - u\|_\infty \to 0$*, as* $n \to \infty$*.*

In the sequel, X will denote the space $E_0^\alpha \times E_0^\beta$*, which is a reflexive Banach space endowed with the norm*

$$\|(u,v)\|_X = \|u\|_\alpha + \|v\|_\beta, \tag{2.10}$$

where $\|u\|_\alpha$ *and* $\|v\|_\beta$ *are defined in (2.5) and (2.9), respectivelly. Obviously, X is compactly embedded in* $C^0([0,T],\mathbf{R}) \times C^0([0,T],\mathbf{R})$*.*

Definition 2.6 By the weak solution of problem (1.1), we mean any $(u,v) \in X$ such that

$$\int_0^T a(t){}_0D_t^\alpha u(t){}_0D_t^\alpha x(t)dt + \int_0^T b(t){}_0D_t^\beta v(t){}_0D_t^\beta y(t)dt - \int_0^T h_1(u(t))x(t)dt$$
$$- \int_0^T h_2(v(t))y(t)dt - \lambda \int_0^T \left(F_u(t,u(t),v(t))x(t) + F_v(t,u(t),v(t))y(t)\right)dt = 0$$

for every $(x,y) \in X$.

We define

$$H_i(x) = \int_0^x h_i(z)dz, \ \text{ and } \ \Theta_i(x) = \int_0^T H_i(x(s))ds \quad i = 1,2 \tag{2.11}$$

for every $t \in [0,T]$ and $x \in \mathbf{R}$.

Lemma 2.7 *Assume that* $h_1, h_2 : \mathbf{R} \to \mathbf{R}$ *satisfy (1.2) and* $H_i(x), \Theta_i(x), (i = 1,2)$ *defined by (2.11). Then the functional* $\Theta(u,v) : X \to \mathbf{R}$ *defined by*

$$\Theta(u,v) := \Theta_1(u) + \Theta_2(v) = \int_0^T H_1(u(t))dt + \int_0^T H_2(v(t))dt \tag{2.12}$$

is a Gâteaux differentiable sequentially weakly continuous functional on X with compact derivative

$$\Theta'(u,v)(x,y) = \int_0^T h_1(u(t))x(t)dt + \int_0^T h_2(v(t))y(t)dt$$

for every $(x, y) \in X$.

Proof Suppose that $\{(u_n, v_n)\} \subset X,\ (u_n, v_n) \rightharpoonup (u, v)$ in X as $n \to +\infty$. It follows from Lemma 2.5 that (u_n, v_n) converges uinformly to (u, v) on $[0, T]$. Thus, there exist constants $c_1, c_2 > 0$ such that $\|u_n\|_\infty \le c_1$ and $\|v_n\|_\infty \le c_2$ for any $n \in \mathbf{N}$. Then

$$\begin{aligned}|H_1(u_n(t)) - H_1(u(t))| &\le L_1 \left| \int_{u(t)}^{u_n(t)} |s| ds \right| \le \frac{L_1}{2}(|u_n(t)|^2 + |u(t)|^2)\\ &\le \frac{L_1}{2}(c_1^2 + \|u\|_\infty^2),\end{aligned}$$

and

$$\begin{aligned}|H_2(v_n(t)) - H_2(v(t))| &\le L_2 \left| \int_{v(t)}^{v_n(t)} |s| ds \right| \le \frac{L_2}{2}(|v_n(t)|^2 + |v(t)|^2)\\ &\le \frac{L_2}{2}(c_2^2 + \|v\|_\infty^2)\end{aligned}$$

for any $n \in \mathbf{N}$ and $t \in [0, T]$. Furthermore, $H_1(u_n(t)) \to H_1(u(t))$ and $H_2(v_n(t)) \to H_2(v(t))$ at every $t \in [0, T]$, and by the Lebesgue Convergence Theorem

$$\begin{aligned}\Theta(u_n, v_n) &= \int_0^T H_1(u_n(t))dt + \int_0^T H_2(v_n(t))dt \to \int_0^T H_1(u(t))dt\\ &\quad + \int_0^T H_2(v(t))dt = \Theta(u, v).\end{aligned}$$

Next we show Gâteaux differentiability of Θ. Suppose $u, x \in E_0^\alpha$ and $s \neq 0$ then

$$\begin{aligned}&\left| \frac{\Theta_1(u + sx) - \Theta_1(u)}{s} - \int_0^T h_1(u(t))x(t)dt \right|\\ &\quad\le \int_0^T \left| \frac{H_1(u + sx) - H_1(u)}{s} - h_1(u(t))x(t) \right| dt\\ &\quad= \int_0^T |h_1(u(t) + s\zeta(t)x(t)) - h_1(u(t))||x(t)|dt\\ &\quad\le L_1 \|x\|_\infty^2 |s|,\end{aligned}$$

where $0 < \zeta(t) < 1$ for any $t \in [0, T]$. Therefore, $\Theta_1 : E_0^\alpha \to \mathbf{R}$ is a Gâteaux differentiable at any $u \in E_0^\alpha$.

Analogously, we have that $\Theta_2 : E_0^\beta \to \mathbf{R}$ is a Gâteaux differentiable at any $v \in E_0^\beta$. Hence, $\Theta : X \to \mathbf{R}$ is a Gâteaux differentiable at any $(u, v) \in X$ with derivative

$$\Theta'(u, v)(x, y) = \int_0^T h_1(u(t))x(t)dt + \int_0^T h_2(v(t))y(t)dt$$

for every $(x, y) \in X$.

For any three elements (u_1, v_1), (u_2, v_2) and (x, y) of X, it is easy to see that

$$\begin{aligned}(\Theta'(u_1, v_1) - \Theta'(u_2, v_2))(x, y) &= \int_0^T (h_1(u_1) - h_1(u_2))x(t)dt \\ &\quad + \int_0^T (h_2(v_1) - h_2(v_2))y(t)dt \\ &\le L_1 \int_0^T |u_1 - u_2||x(t)|dt + L_2 \int_0^T |v_1 - v_2||y(t)|dt \\ &\le \frac{L_1 T^{\alpha+\frac{1}{2}}}{\Gamma(\alpha)\sqrt{a_0(2\alpha - 1)}} \|u_1 - u_2\|_\infty \|x\|_\alpha \\ &\quad + \frac{L_2 T^{\beta+\frac{1}{2}}}{\Gamma(\beta)\sqrt{b_0(2\beta - 1)}} \|v_1 - v_2\|_\infty \|y\|_\beta,\end{aligned}$$

which implies

$$\|\Theta'(u_1, v_1) - \Theta'(u_2, v_2)\|_X \le T^*(\|u_1 - u_2\|_\infty + \|v_1 - v_2\|_\infty),$$

where

$$T^* := \max\left\{\frac{L_1 T^{\alpha+\frac{1}{2}}}{\Gamma(\alpha)\sqrt{a_0(2\alpha - 1)}}, \frac{L_2 T^{\beta+\frac{1}{2}}}{\Gamma(\beta)\sqrt{b_0(2\beta - 1)}}\right\}.$$

Hence $\Theta' : X \to X^*$ is a compact operator.

Similarly to the proof of [16, Theorem 5.1], we have □

Lemma 2.8 *Let $\frac{1}{2} < \alpha, \beta \le 1$ and $(u, v) \in X$. If (u, v) is a non-trivial weak solution of problem (1.1), then (u, v) is also a non-trivial solution of problem (1.1).*

Our analysis is mainly based on the following critical points theorem of Bonanno and Molica Bisci [21], which is a more precise result of Ricceri [20, Theorem 2.5]:

Lemma 2.9 *([21, Theorem 2.1]). Let X be a reflexive real Banach space. Let $\Phi, \Psi : X \to \mathbf{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then,

(1) *If* $\gamma < +\infty$ *and* $\lambda \in]0, \frac{1}{\gamma}[$, *the following alternative holds: either the functional* $\Phi - \lambda\Psi$ *has a global minimum, or there exists a sequence* $\{u_n\}$ *of local minima of* $\Phi - \lambda\Psi$ *such that* $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(2) *If* $\delta < +\infty$ *and* $\lambda \in]0, \frac{1}{\delta}[$, *the following alternative holds: either there exists a global minimum of* Φ *which is a local minimum of* $\Phi - \lambda\Psi$, *or there exists a sequence* $\{u_n\}$ *of pairwise distinct local minima of* $\Phi - \lambda\Psi$, *with* $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, *which weakly converges to a global minimum of* Φ.

3 Main results and proof

In this section, we will state and prove our main results. For convenience, put

$$\kappa := \min\left\{1 - \frac{L_1 T^{2\alpha}}{(\Gamma(\alpha+1))^2 a_0},\ 1 - \frac{L_2 T^{2\beta}}{(\Gamma(\beta+1))^2 b_0}\right\}, \tag{3.1}$$

$$\rho := \max\left\{1 + \frac{L_1 T^{2\alpha}}{(\Gamma(\alpha+1))^2 a_0},\ 1 + \frac{L_2 T^{2\beta}}{(\Gamma(\beta+1))^2 b_0}\right\}. \tag{3.2}$$

For a given constant $\theta \in (0, \frac{1}{2})$, set

$$\begin{aligned} P(\alpha,\theta) = \frac{1}{2\theta^2 T^2}\Bigg\{&\int_0^T a(t)t^{2(1-\alpha)}dt + \int_{\theta T}^T a(t)(t-\theta T)^{2(1-\alpha)}dt \\ &+ \int_{(1-\theta)T}^T a(t)(t-(1-\theta)T)^{2(1-\alpha)}dt - 2\int_{(1-\theta)T}^T a(t)(t^2-(1-\theta)Tt)^{1-\alpha}dt \\ &- 2\int_{\theta T}^T a(t)(t^2-\theta Tt)^{1-\alpha}dt + 2\int_{(1-\theta)T}^T a(t)(t^2-\theta Tt+\theta(1-\theta)T^2)^{1-\alpha}dt\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} Q(\beta,\theta) = \frac{1}{2\theta^2 T^2}\Bigg\{&\int_0^T b(t)t^{2(1-\beta)}dt + \int_{\theta T}^T b(t)(t-\theta T)^{2(1-\beta)}dt \\ &+ \int_{(1-\theta)T}^T b(t)(t-(1-\theta)T)^{2(1-\beta)}dt - 2\int_{(1-\theta)T}^T b(t)(t^2-(1-\theta)Tt)^{1-\beta}dt \\ &- 2\int_{\theta T}^T b(t)(t^2-\theta Tt)^{1-\beta}dt + 2\int_{(1-\theta)T}^T b(t)(t^2-Tt+\theta(1-\theta)T^2)^{1-\beta}dt\Bigg\}. \end{aligned}$$

For any $d > 0$ we denote by $\Omega(d)$ the set

$$\left\{(x, y) \in \mathbf{R^2} : \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 \leq d\right\}. \tag{3.3}$$

Theorem 3.1 *Assume that* $\kappa > 0$ *and (H0), (H1) hold. Furthermore*

(**H2**) $F(t, x, y) \geq 0$ *for all* $(t, x, y) \in [0, T] \times [0, +\infty) \times [0, +\infty)$;

(**H3**) *There exists* $\theta \in (0, \frac{1}{2})$ *such that, if we put*

$$A_\infty := \liminf_{\xi \to +\infty} \frac{\int_0^T \sup_{|x|+|y| \leq \xi} F(t, x, y)dt}{\xi^2}, \quad \text{and}$$

$$B_\infty := \limsup_{\xi \to +\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\xi, \Gamma(2-\beta)\xi)dt}{\xi^2},$$

one has

$$A_\infty < \frac{\kappa}{8M\rho\Delta} B_\infty. \tag{3.4}$$

where $\Delta := \max\{P(\alpha, \theta), \quad Q(\beta, \theta)\}$ *and* M *is given in (1.5). Then, for every*

$$\lambda \in \Lambda := \left]\frac{\rho\Delta}{B_\infty}, \frac{\kappa}{8MA_\infty}\right[,$$

problem (1.1) admits an unbounded sequence of weak solutions in X.

Proof Our aim is to apply part (1) of Lemma 2.9 to problem (1.1). We begin by taking $X = E_0^\alpha \times E_0^\beta$ endowed with the norm $\|(u, v)\|_X$ as considered in (2.10). Define the functional $I_\lambda : X \to \mathbf{R}$ given by

$$I_\lambda(u, v) = \Phi(u, v) - \lambda\Psi(u, v)$$

for all $(u, v) \in X$, where

$$\Phi(u, v) = \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\|v\|_\beta^2 - \Theta(u, v), \tag{3.5}$$

and

$$\Psi(u, v) = \int_0^T F(t, u(t), v(t))dt. \tag{3.6}$$

Since X is compactly embedded in $C^0([0, T], \mathbf{R}) \times C^0([0, T], \mathbf{R})$, it is well known that Ψ is well-defined Gâteaux differentiable functional whose Gâteaux derivative at the point $(u, v) \in X$ is the functional $\Psi'(u, v) \in X^*$, given by

$$\Psi'(u, v)(x, y) = \int_0^T (F_u(t, u(t), v(t))x(t) + F_v(t, u(t), v(t))y(t))\, dt$$

for every $(x, y) \in X$.

We claim that the functional Ψ is a sequentially weakly upper semicontinuous functional on X. Indeed, for fixed $(u, v) \in X$, suppose that $\{(u_n, v_n)\} \subset X$, $(u_n, v_n) \rightharpoonup (u, v)$ in X as $n \to +\infty$. Then (u_n, v_n) converges uinformly to (u, v) on $[0, T]$. Hence

$$\limsup_{n\to+\infty} \Psi(u_n, v_n) \le \int_0^T \limsup_{n\to+\infty} F(t, u_n, v_n)dt = \int_0^T F(t, u, v)dt = \Psi(u, v),$$

which implies that Ψ is sequentially weakly upper semicontinuous. Hence the claim is true.

As concerns the functional Φ, we can show that Φ defined by (3.5) is a sequentially weakly lower semicontinuous, strongly continuous, and coercive functional on X. In fact, since (1.2) holds for every $x_1, x_2 \in \mathbf{R}$ and $h_1(0) = h_2(0) = 0$, one has $|h_i(x)| \le L_i|x|, i = 1, 2$ for all $x \in \mathbf{R}$. It follows from (2.3), (2.7) and Lemma 2.5 that

$$\begin{aligned}
\Phi(u, v) &\ge \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\|v\|_\beta^2 - \left|\int_0^T H_1(u(t))dt\right| - \left|\int_0^T H_2(v(t))dt\right| \\
&\ge \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\|v\|_\beta^2 - \frac{L_1}{2}\int_0^T |u(t)|^2 dt - \frac{L_2}{2}\int_0^T |v(t)|^2 dt \\
&\ge \left(\frac{1}{2} - \frac{L_1 T^{2\alpha}}{2(\Gamma(\alpha+1))^2 a_0}\right)\|u\|_\alpha^2 + \left(\frac{1}{2} - \frac{L_2 T^{2\beta}}{2(\Gamma(\beta+1))^2 b_0}\right)\|v\|_\beta^2 \\
&\ge \frac{\kappa}{2}(\|u\|_\alpha^2 + \|v\|_\beta^2), \qquad (3.7)
\end{aligned}$$

for all $(u, v) \in X$ and similarly

$$\begin{aligned}
\Phi(u, v) &\le \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\|v\|_\beta^2 + \left|\int_0^T H_1(u(t))dt\right| + \left|\int_0^T H_2(v(t))dt\right| \\
&\le \left(\frac{1}{2} + \frac{L_1 T^{2\alpha}}{2(\Gamma(\alpha+1))^2 a_0}\right)\|u\|_\alpha^2 + \left(\frac{1}{2} + \frac{L_2 T^{2\beta}}{2(\Gamma(\beta+1))^2 b_0}\right)\|v\|_\beta^2 \\
&\le \frac{\rho}{2}(\|u\|_\alpha^2 + \|v\|_\beta^2). \qquad (3.8)
\end{aligned}$$

for all $(u, v) \in X$. So Φ is coercive. □

Moreover, $\Phi + \Theta$ is a continuous functional on X and Θ, from Lemma 2.5, is a Gâteaux differentiable sequentially weakly continuous and therefore continuous on X, then Φ is a continuous functional on X. It is not difficult to verify that the functional Φ is a Gâteaux differentiable functional with the differential

$$\Phi'(u,v)(x,y) = \int_0^T a(t)_0D_t^\alpha u(t)_0D_t^\alpha x(t)dt + \int_0^T b(t)_0D_t^\beta v(t)_0D_t^\beta y(t)dt - \int_0^T h_1(u(t))x(t)dt - \int_0^T h_2(v(t))y(t)dt.$$

Furthermore, Φ is also sequentially weakly lower semicontinuous on X since Θ is sequentially weakly lower semicontinuous, and if $(u_n, v_n) \rightharpoonup (u, v)$ in X then

$$\liminf_{n\to\infty} \Phi(u_n, v_n) = \liminf_{n\to\infty}\left(\frac{1}{2}\|u_n\|_\alpha^2 + \frac{1}{2}\|v_n\|_\beta^2\right) - \lim_{n\to\infty}\Theta(u_n, v_n) \geq \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\|v\|_\beta^2 - \Theta(u,v) = \Phi(u,v).$$

It is easy to show that the critical points of the functional I_λ and the weak solutions of the problem (1.1) are the same and by Lemma 2.9 we prove our result.

According to (2.1), taking (2.4) and (2.8) into account, one has

$$\max_{t\in[0,T]} |u(t)|^2 \leq M\|u\|_\alpha^2 \quad \text{and} \quad \max_{t\in[0,T]} |v(t)|^2 \leq M\|v\|_\beta^2$$

for every $(u, v) \in X$.

Hence

$$\max_{t\in[0,T]} (|u(t)|^2 + |v(t)|^2) \leq M(\|u\|_\alpha^2 + \|v\|_\beta^2).$$

So, for every $r > 0$, from the definition of Φ and by using (3.7) one has

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) :&= \{(u,v) \in X : \ \Phi(u,v) \leq r\} \\ &\subseteq \left\{(u,v) \in X : \ \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\|v\|_\beta^2 \leq \frac{r}{\kappa}\right\} \\ &\subseteq \left\{(u,v) \in X : \ \frac{(\Gamma(\alpha))^2 a_0(2\alpha-1)}{2T^{2\alpha-1}}\|u\|_\infty^2 \right. \\ &\qquad \left. + \frac{(\Gamma(\beta))^2 b_0(2\beta-1)}{2T^{2\beta-1}}\|v\|_\infty^2 \leq \frac{r}{\kappa}\right\} \\ &\subseteq \left\{(u,v) \in X : \ \frac{1}{2}|u(t)|^2 + \frac{1}{2}|v(t)|^2 \leq \frac{Mr}{\kappa}, \ \text{for all} \ t \in [0,T]\right\}. \end{aligned} \tag{3.9}$$

Set

$$\varphi(r) := \inf_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{(x,y)\in\Phi^{-1}(]-\infty,r[)}\Psi(x,y)\right) - \Psi(u,v)}{r - \Phi(u,v)}.$$

Note that $\Phi(0,0)=0$, and from the condition (H1), $\Psi(0,0)\geq 0$. Hence, for every $r>0$,

$$\begin{aligned}
\varphi(r) &= \inf_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{(x,y)\in\Phi^{-1}(]-\infty,r[)}\Psi(x,y)\right)-\Psi(u,v)}{r-\Phi(u,v)}\\
&\leq \frac{\sup_{(x,y)\in\Phi^{-1}(]-\infty,r[)}\Psi(x,y)}{r}\\
&= \frac{\sup_{\Phi(u,v)<r}\int_0^T F(t,u,v)dt}{r},
\end{aligned}$$

and it follows from (3.9) that

$$\varphi(r)\leq \frac{1}{r}\sup_{\Omega\left(\frac{Mr}{\kappa}\right)}\int_0^T F(t,u,v)dt$$

where $\Omega\left(\frac{Mr}{\kappa}\right):=\{(u,v)\in X:\ \frac{1}{2}|u(t)|^2+\frac{1}{2}|v(t)|^2<\frac{Mr}{\kappa}, \forall t\in[0,T]\}$.

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n\to+\infty$ and

$$\lim_{n\to+\infty}\frac{\int_0^T\sup_{|x|+|y|<\xi_n}F(t,x,y)dt}{\xi_n^2}=A_\infty<+\infty.$$

Put $r_n:=\frac{\kappa}{8M}\xi_n^2$ for all $n\in\mathbf{N}$. Let $(u,v)\in\Phi^{-1}(]-\infty,r_n[)$, by (3.9) one has

$$\frac{1}{2}|u(t)|^2+\frac{1}{2}|v(t)|^2\leq\frac{M}{\kappa}r_n,\quad \forall t\in[0,T],$$

which implies

$$|u(t)|\leq\sqrt{\frac{2Mr_n}{\kappa}}\ \text{ and }\ |v(t)|\leq\sqrt{\frac{2Mr_n}{\kappa}}.$$

Hence, for n large enough ($r_n>1$)

$$|u(t)|+|v(t)|\leq 2\sqrt{\frac{2Mr_n}{\kappa}}=\xi_n.$$

Thus, for all $n\in\mathbf{N}$

$$\begin{aligned}
\varphi(r_n) &= \frac{8M}{\kappa\xi_n^2}\cdot\sup_{\{(u,v)\in X:\ |u(t)|+|v(t)|<\xi_n,\forall t\in[0,T]\}}\int_0^T F(t,u,v)dt\\
&\leq\frac{8M}{\kappa}\cdot\frac{\int_0^T\sup_{|x|+|y|<\xi_n}F(t,x,y)dt}{\xi_n^2}.
\end{aligned}$$

Let

$$\gamma := \liminf_{r\to+\infty} \varphi(r).$$

Then

$$\begin{aligned}\gamma &\le \liminf_{n\to+\infty} \varphi(r_n)\\ &\le \frac{8M}{\kappa}\cdot \lim_{n\to+\infty} \frac{\int_0^T \sup_{|x|+|y|<\xi_n} F(t,x,y)dt}{\xi_n^2}\\ &= \frac{8M}{\kappa}A_\infty < +\infty.\end{aligned}$$

Hence, $\Lambda \subseteq]0, \frac{1}{\gamma}[$.

For $\lambda \in \Lambda$, we shall show that the functional I_λ is unbounded from below. Indeed, since $\frac{B_\infty}{\rho\Delta} > \frac{1}{\lambda}$, we can choose a sequence $\{\eta_n\}$ of positive numbers and $\varepsilon > 0$ such that $\eta_n \to +\infty$ and

$$\frac{1}{\lambda} < \varepsilon < \frac{1}{\rho\Delta}\cdot \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\eta_n, \Gamma(2-\beta)\eta_n)dt}{\eta_n^2}, \tag{3.10}$$

for n large enough.

For all $n \in \mathbf{N}$, and $\theta \in \left(0, \frac{1}{2}\right)$ define $\omega_n = (\omega_{1,n}(t), \omega_{2,n}(t))$ by setting

$$\omega_{1,n}(t) = \begin{cases} \frac{\Gamma(2-\alpha)\eta_n}{\theta T} t, & t \in [0, \theta T[,\\ \Gamma(2-\alpha)\eta_n, & t \in [\theta T, (1-\theta)T],\\ \frac{\Gamma(2-\alpha)\eta_n}{\theta T}(T-t), & t \in](1-\theta)T, T] \end{cases} \tag{3.11}$$

and

$$\omega_{2,n}(t) = \begin{cases} \frac{\Gamma(2-\beta)\eta_n}{\theta T} t, & t \in [0, \theta T[,\\ \Gamma(2-\beta)\eta_n, & t \in [\theta T, (1-\theta)T],\\ \frac{\Gamma(2-\beta)\eta_n}{\theta T}(T-t), & t \in](1-\theta)T, T]. \end{cases} \tag{3.12}$$

Clearly $\omega_{i,n}(0) = \omega_{i,n}(T) = 0$ and $\omega_{i,n} \in L^2[0, T]$ for $i = 1, 2$. A direct calculation shows that

$${}_0D_t^\alpha \omega_{1,n}(t) = \begin{cases} \frac{\eta_n}{\theta T} t^{1-\alpha}, & t \in [0, \theta T[,\\ \frac{\eta_n}{\theta T}(t^{1-\alpha} - (t-\theta T)^{1-\alpha}), & t \in [\theta T, (1-\theta)T],\\ \frac{\eta_n}{\theta T}(t^{1-\alpha} - (t-\theta T)^{1-\alpha} - (t-(1-\theta)T)^{1-\alpha}), & t \in](1-\theta)T, T] \end{cases}$$

and

$$
{}_0D_t^{\beta}\omega_{2,n}(t)=\begin{cases}\frac{\eta_n}{\theta T}t^{1-\beta}, & t\in[0,\theta T[,\\ \frac{\eta_n}{\theta T}(t^{1-\beta}-(t-\theta T)^{1-\beta}), & t\in[\theta T,(1-\theta)T],\\ \frac{\eta_n}{\theta T}(t^{1-\beta}-(t-\theta T)^{1-\beta}-(t-(1-\theta)T)^{1-\beta}), & t\in](1-\theta)T,T].\end{cases}
$$

Furthermore,

$$
\begin{aligned}
\int_0^T a(t)|{}_0D_t^{\alpha}\omega_{1,n}(t)|^2dt &= \int_0^{\theta T}+\int_{\theta T}^{(1-\theta)T}+\int_{(1-\theta)T}^{T}(a(t)|{}_0D_t^{\alpha}\omega_{1,n}(t)|^2dt\\
&= \frac{\eta_n^2}{\theta^2T^2}\left\{\int_0^T a(t)t^{2(1-\alpha)}dt+\int_{\theta T}^T a(t)(t-\theta T)^{2(1-\alpha)}dt\right.\\
&\quad+\int_{(1-\theta)T}^T a(t)(t-(1-\theta)T)^{2(1-\alpha)}dt\\
&\quad-2\int_{\theta T}^T a(t)(t^2-\theta Tt)^{1-\alpha}dt\\
&\quad-2\int_{(1-\theta)T}^T a(t)(t^2-(1-\theta)Tt)^{1-\alpha}dt\\
&\quad\left.+2\int_{(1-\theta)T}^T a(t)(t^2-\theta Tt+\theta(1-\theta)T^2)^{1-\alpha}dt\right\}\\
&= 2P(\alpha,\theta)\eta_n^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T b(t)|{}_0D_t^{\beta}\omega_{2,n}(t)|^2dt &= \int_0^{\theta T}+\int_{\theta T}^{(1-\theta)T}+\int_{(1-\theta)T}^{T}b(t)|{}_0D_t^{\beta}\omega_{2,n}(t)|^2dt\\
&= \frac{\eta_n^2}{\theta^2T^2}\left\{\int_0^T b(t)t^{2(1-\beta)}dt+\int_{\theta T}^T b(t)(t-\theta T)^{2(1-\beta)}dt\right.\\
&\quad+\int_{(1-\theta)T}^T b(t)(t-(1-\theta)T)^{2(1-\beta)}dt\\
&\quad-2\int_{\theta T}^T b(t)(t^2-\theta Tt)^{1-\beta}dt\\
&\quad-2\int_{(1-\theta)T}^T b(t)(t^2-(1-\theta)Tt)^{1-\beta}dt\\
&\quad\left.+2\int_{(1-\theta)T}^T b(t)(t^2-Tt+\theta(1-\theta)T^2)^{1-\beta}dt\right\}\\
&= 2Q(\beta,\theta)\eta_n^2.
\end{aligned}
$$

Thus, $\omega_n \in X$, and

$$\|\omega_{1,n}\|_\alpha^2 = \int_0^T a(t)|_0 D_t^\alpha \omega_{1,n}(t)|^2 dt = 2P(\alpha, \theta)\eta_n^2,$$
$$\|\omega_{2,n}\|_\beta^2 = \int_0^T b(t)|_0 D_t^\beta \omega_{2,n}(t)|^2 dt = 2Q(\beta, \theta)\eta_n^2.$$

This and (3.8) imply that

$$\begin{aligned}
\Phi(\omega_{1,n}, \omega_{2,n}) &= \frac{1}{2}\|\omega_{1,n}\|_\alpha^2 + \frac{1}{2}\|\omega_{2,n}\|_\beta^2 - \Theta(\omega_{1,n}, \omega_{2,n}) \\
&\leq \frac{\rho}{2}(\|\omega_{1,n}\|_\alpha^2 + \|\omega_{2,n}\|_\beta^2) \\
&= \rho(P(\alpha, \theta) + Q(\beta, \theta))\eta_n^2 \leq \rho\Delta\eta_n^2. \qquad (3.13)
\end{aligned}$$

From (H2), we have

$$\begin{aligned}
\Psi(\omega_{1,n}, \omega_{2,n}) &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^T F(t, \omega_{1,n}, \omega_{2,n})dt \\
&\geq \int_{\theta T}^{(1-\theta)T} F(t, \omega_{1,n}, \omega_{2,n})dt \\
&= \int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\eta_n, \Gamma(2-\beta)\eta_n)dt. \qquad (3.14)
\end{aligned}$$

According to (3.10), (3.13) and (3.14), we have

$$\begin{aligned}
I_\lambda(\omega_{1,n}, \omega_{2,n}) &= \Phi(\omega_{1,n}, \omega_{2,n}) - \lambda\Psi(\omega_{1,n}, \omega_{2,n}) \\
&\leq \rho(P(\alpha, \theta) + Q(\beta, \theta))\eta_n^2 - \lambda \int_{\theta T}^{(1-\theta)T} \\
&\quad \times F(t, \Gamma(2-\alpha)\eta_n, \Gamma(2-\beta)\eta_n)dt \\
&\leq \rho\Delta(1 - \lambda\varepsilon)\eta_n^2.
\end{aligned}$$

for n large enough. Taking into account the choice of ε, the above inequality shows that

$$\lim_{n\to+\infty} I_\lambda(\omega_{1,n}, \omega_{2,n}) = -\infty,$$

which implies the functional I_λ is unbounded from below and the claim follows.

By using the case (1) of Lemma 2.9, the functional I_λ has a sequence $\{(u_n, v_n)\}$ of critical points such that $\Phi(u_n, v_n) \to +\infty$. From (2.10) and (3.8), we get $\|(u_n, v_n)\|_X \geq \sqrt{\frac{2\Phi(u_n, v_n)}{\rho}}$, which implies $\|(u_n, v_n)\|_X \to +\infty$, and the proof of Theorem 3.1 is complete.

Theorem 3.2 *Assume that* $\kappa > 0$ *and (H0), (H2) hold. Furthermore*

(**H4**) $F(t, 0, 0) = 0$ *for all* $t \in [0, T]$.

(**H5**) *There exists* $\theta \in (0, \frac{1}{2})$ *such that, if we put*

$$A_0 := \liminf_{\xi \to 0^+} \frac{\int_0^T \sup_{|x|+|y| \le \xi} F(t, x, y) dt}{\xi^2} \quad \text{and}$$

$$B_0 := \limsup_{\xi \to 0^+} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\xi, \Gamma(2-\beta)\xi) dt}{\xi^2},$$

one has

$$A_0 < \frac{\kappa}{8M\rho\Delta} B_0. \tag{3.15}$$

where $\Delta = \max\{P(\alpha, \theta), \quad Q(\beta, \theta)\}$ *and* M *is given in (1.5).*

Then, for every

$$\lambda \in \Lambda' := \left] \frac{\rho\Delta}{B_0}, \ \frac{\kappa}{8MA_0} \right[,$$

problem (1.1) admits a sequence $\{(u_n, v_n)\}$ *of weak solutions such that* $(u_n, v_n) \rightharpoonup (0, 0)$.

Proof Our goal is to apply part (2) of Lemma 2.9 to Φ and Ψ defined in (3.3) and (3.4) respectively. As it has been pointed out before, the functionals Φ, Ψ satisfy the assumptions regularity required in Lemma 2.9.

Since $F(t, 0, 0) = 0$ for all $t \in [0, T]$. Then $\min_{(u,v)\in X} \Phi(u, v) = \Phi(0, 0) = 0$. Let $\{\zeta_n\}$ be a sequence of positive numbers such that $\zeta_n \to 0$ and

$$\lim_{n \to +\infty} \frac{\int_0^T \sup_{|x|+|y|<\zeta_n} F(t, x, y) dt}{\zeta_n^2} = A_0 < +\infty.$$

Setting $r_n = \frac{\kappa}{8M}\zeta_n^2$ for all $n \in \mathbf{N}$, and working as in the proof of Theorem 3.1, we can show that

$$\delta = \liminf_{r \to (\inf_X \Phi)^+} \varphi(r) \le \frac{8M}{\kappa} \cdot \lim_{n \to +\infty} \frac{\int_0^T \sup_{|x|+|y|<\zeta_n} F(t, x, y) dt}{\zeta_n^2} = \frac{8M}{\kappa} A_0,$$

and so $\Lambda' \subset]0, \frac{1}{\delta}[$.

Now fix λ as in the conclusion, then

$$\frac{1}{\lambda} < \frac{1}{\rho\Delta} \limsup_{\xi \to 0^+} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\xi, \Gamma(2-\beta)\xi) dt}{\xi^2},$$

and there exist a sequence $\{\tau_n\}$ of positive numbers and a constant ε_1 such that $\tau_n \le \frac{1}{n}$ and

$$\lim_{n\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\tau_n, \Gamma(2-\beta)\tau_n)dt}{\tau_n^2} = \limsup_{\xi\to 0^+} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\xi, \Gamma(2-\beta)\xi)dt}{\xi^2},$$

and in addition

$$\frac{1}{\lambda} < \varepsilon_1 < \frac{1}{\rho\Delta} \lim_{n\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\tau_n, \Gamma(2-\beta)\tau_n)dt}{\tau_n^2}.$$

For all $n \in \mathbf{N}$, and $\theta \in \left(0, \frac{1}{2}\right)$ define $\omega_n = (\omega_{1,n}(t), \omega_{2,n}(t))$ by setting

$$\omega_{1,n}(t) = \begin{cases} \frac{\Gamma(2-\alpha)\tau_n}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\alpha)\tau_n, & t \in [\theta T, (1-\theta)T], \\ \frac{\Gamma(2-\alpha)\tau_n}{\theta T}(T-t), & t \in](1-\theta)T, T] \end{cases}$$

and

$$\omega_{2,n}(t) = \begin{cases} \frac{\Gamma(2-\beta)\tau_n}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\beta)\tau_n, & t \in [\theta T, (1-\theta)T], \\ \frac{\Gamma(2-\beta)\tau_n}{\theta T}(T-t), & t \in](1-\theta)T, T]. \end{cases}$$

Clearly $\omega_{i,n}(0) = \omega_{i,n}(T) = 0$ for $i = 1, 2$, and $\{\omega_n\}$ converges strongly to $(0, 0)$ in X. By the same arguing as inside in Theorem 3.1, we have

$$\begin{aligned} I_\lambda(\omega_{1,n}, \omega_{2,n}) &= \Phi(\omega_{1,n}, \omega_{2,n}) - \lambda\Psi(\omega_{1,n}, \omega_{2,n}) \\ &\le \rho(P(\alpha,\theta)+Q(\beta,\theta))\tau_n^2 - \lambda \int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\tau_n, \Gamma(2-\beta)\tau_n)dt \\ &\le \rho\Delta(1-\lambda\varepsilon_1)\tau_n^2 \\ &< 0 = I_\lambda(0, 0). \end{aligned}$$

for n large enough. This together with the fact that $\|\omega_n\|_X = \|(\omega_{1,n}, \omega_{2,n})\|_X \to 0$ shows that I_λ has not a local minimum at zero, and the claim follows.

The alternative of Lemma 2.9 case (2) ensures the existence of sequence $\{(u_n, v_n)\}$ of pairwise distinct local minima of I_λ which weakly converges to $(0, 0)$. This completes the proof of Theorem 3.2. □

Finally, we present an example to illustrate our main results.

Example 3.3 Consider the following fractional differential system:

$$\begin{cases} {}_tD_1^{0.75}((2+t)\cdot {}_0D_t^{0.75}u(t)) = \lambda F_u(t,u(t),v(t)) + \frac{1}{4}\sin u(t), & 0<t<1,\\ {}_tD_1^{0.8}((1+t^3)\cdot {}_0D_t^{0.8}v(t)) = \lambda F_v(t,u(t),v(t)) + \frac{1}{3}\sqrt[3]{v(t)+1}-1, & 0<t<1,\\ u(0)=u(1)=0, \quad v(0)=v(1)=0, \end{cases} \tag{3.16}$$

where $T = 1$, $\alpha = 0.75$, $\beta = 0.8$, $a(t) = 2+t$, $b(t) = 1+t^3$, and $h_1(u) = \frac{1}{4}\sin u(t)$, $h_2(v) = \frac{1}{3}\sqrt[3]{v(t)+1}-1$. Moreover, for all $(t,u,v) \in [0,1]\times \mathbf{R}^2$, put $F(t,u,v) = (1+t^2)H(u,v)$, where

$$H(u,v) = \begin{cases} (\xi_{n+1})^3 e^{-\frac{1}{1-\left[(u-0.9064\xi_{n+1})^2+(v-0.9182\xi_{n+1})^2\right]}}, & (u,v)\in\Omega,\\ 0, \quad (u,v)\in \mathbf{R}^2\setminus\Omega, \end{cases}$$

where

$$\Omega = \bigcup\nolimits_{n\ge 1}\{(u,v) : (u-0.9064\xi_{n+1})^2 + (v-0.9182\xi_{n+1})^2 < 1\},$$

and $\xi_1 = 1$, $\xi_{n+1} = n(\xi_n)^{\frac{5}{3}} + 1$ for all $n \in \mathbf{N}$.

Obviously, $h_1, h_2 : \mathbf{R} \to \mathbf{R}$ are two Lipschitz continuous functions with the Lipschitz constants $L_1 = \frac{1}{4}$, $L_2 = \frac{1}{9}$ and $h_1(0) = h_2(0) = 0$; $F(t,0,0) = 0$ for all $t \in [0,1]$. With the aid of direct computation we have that $a_0 = 2$, $b_0 = 1$, and

$$M \approx 1.2302, \quad \kappa \approx 0.8520, \quad \rho \approx 1.1480.$$

Let $\theta = \frac{1}{4}$, then we have

$$P(\alpha,\theta) = P(0.75, 0.25) \approx 7.9576, \qquad Q(\beta,\theta) = Q(0.8, 0.25) \approx 4.4641,$$

hence $\Delta = 7.9576$. Then all conditions of Thorem 3.1 are satisfid. In fact, the conditions (H0), (H1) and (H2) hold. For every $n \in \mathbf{N}$, the restriction of $H(u,v)$ on Ω attains its maximum in $(0.9064\xi_{n+1}, 0.9182\xi_{n+1})$ and

$$H(0.9064\xi_{n+1}, 0.9182\xi_{n+1}) = (\xi_{n+1})^3 e^{-1}.$$

Moreover,

$$\sup_{|u|+|v|\le 0.9064\xi_{n+1}-1} H(u,v) = (\xi_n)^3 e^{-1},$$

and so

$$
\begin{aligned}
B_\infty &= \lim_{n\to+\infty}\sup \frac{\int_{1/4}^{3/4}(1+t^2)H(0.9064\xi_{n+1}, 0.9182\xi_{n+1})dt}{(\xi_{n+1})^2}\\
&= \frac{96}{61}\cdot\lim_{n\to+\infty}\frac{(\xi_{n+1})^3e^{-1}}{(\xi_{n+1})^2} = +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
A_\infty &= \liminf_{n\to+\infty}\frac{\int_0^T(1+t^2)\sup\limits_{|u|+|v|\le 0.9064\xi_{n+1}-1}H(u,v)dt}{(0.9064\xi_{n+1}-1)^2}\\
&= \frac{4}{3}\cdot\lim_{n\to+\infty}\frac{(\xi_n)^3e^{-1}}{(0.9064\xi_{n+1}-1)^2}\\
&= 0 < \frac{\kappa}{8M\rho\Delta}B_\infty,
\end{aligned}
$$

which implies that the condition (H3) holds. Hence, owing to Theorem 3.1, for each $\lambda \in]0, +\infty[$, the coupled system (3.16) admits an unbounded sequence of weak solutions.

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